VIII. *On the Theory of the Moon. By* John William Lubbock, *Esq. V.P. and Treas. R.S.*

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When I commenced the investigations relating to the theory of the moon which I have had the honour to communicate to the Society, I proposed to show how, by a different but more direct method, the numerical results given by M. Damoiseau might be obtained. The approximations were in fact carried much further by M. Damoiseau than had been done before, and the details which accompany M. Damoiseau's work evince at once the immense labour of the undertaking, and inspire confidence in the accuracy of the results offered. But the state of the question is now changed by the appearance of M. Plana's admirable work, entitled "Théorie du Mouvement de la Lune," in which, although M. Plana employs the same differential equations as those used by M. Damoiseau, and obtains in the same manner finally the expressions for the coordinates of the moon, in terms of the mean longitude by the reversion of series, yet M. Plana's expressions have a very different analytical character and importance, from the circumstance that the author develops all the quantities introduced by integration, according to powers of the quantity called m, which expresses the ratio of the sun's mean motion to that of the moon. In this form of the expression the coefficients of the different powers of m, of the eccentricity, &c., are determinate, as are, for example, the numerical coefficients in the expression for the sine in terms of the arc, and other similar series. An inestimable advantage results from this procedure, which more than compensates for the great increase of labour it occasions, by diminishing the danger of neglecting any terms of the same order as those taken into account, and by affording the means of verifying many terms long before final and complete results shall have been obtained independently by myself or any other person. By treating the differential equations in which the time is the independent variable, as I have proposed, similar results to those of M. Plana may be obtained directly; but the calculations which are required in either method are so prodigiously irksome and laborious, that until identical expressions have actually been obtained independently, to the extent of every sensible term, the theory of the moon cannot, I think, be considered complete. It might, indeed, be supposed that already, through the labours of mathematicians, from Clairaut to the present time, the numerical values of the coefficients of the different inequalities were ascertained with sufficient accuracy for practical purposes, and that any further researches connected with the subject would be more likely to gratify curiosity than to lead to any useful result.

Astronomical observations are now made with so great precision, that the numerical values of the coefficients are wanted to at least the tenth of a second of space: very few, however, of the coefficients of MM. DAMOISEAU and PLANA agree so nearly, and some differ much more, as may be seen in the following comparison of the numerical values of the coefficients of some of the arguments in the expression for the true longitude of the moon in terms of her mean longitude, being indeed those which differ the most.

	Argument.	DAMOISEAU.	PLANA.
1	2τ	+ 2370·00	+ 2370·320
2	ξ	+22639·70	+22641·626
3	$2\tau-\xi$	+ 4589·61	+ 4585·648
4	$2\tau+\xi$	+ 192·22	+ 192·146
5	$\xi_{,}$	− 673·70	− 668·644
6	$2\tau-\xi_{,}$	+ 165·56	+ 165·850
7	$2\tau+\xi_{,}$	− 24·82	− 23·611
8	2ξ	+ 768·72	+ 769·477
9	$2\tau-2\xi$	+ 211·57	+ 212·363
10	$2\tau+2\xi$	+ 14·74	+ 14·119
11	$\xi+\xi_{,}$	− 109·27	− 111·099
12	$2\tau-\xi-\xi_{,}$	+ 207·09	+ 209·742
22	$2\tau+3\xi$	+ 1·27	+ 3·309
24	$2\tau-2\xi-\xi_{,}$	+ 8·99	+ 7·762
27	$2\tau-2\xi+\xi_{,}$	+ 2·55	− 1·395
64	$2\tau+2\eta$	− 5·75	− 3·376
65	$\xi-2\eta$	+ 39·51	+ 37·191
110	$\tau-\xi+\xi_{,}$	+ 2·05	+ ·466
136	$4\tau-2\xi$	+ 31·19	+ 34·518
	$4\tau-2\xi-\xi_{,}$	+ 3·05	+ 1·197

When the coefficients of the inequalities have been determined analytically, it remains to determine with corresponding precision the numerical values of the arbitrary quantities m, e, and γ. The quantity m is already accurately known, but the quantities e and γ must be obtained from the coefficients of $\sin\xi$ in the expression for the longitude, and of $\sin\eta$ in the expression for the latitude, by the reversion of series; and it seems to me that the manner in which these arbitrary quantities are to be determined must be carefully and rigorously defined.

I propose to obtain the expression for the radius vector by means of the equation,

$$\frac{\mathrm{d}^2 . r^2}{2\,\mathrm{d}t^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int \mathrm{d}R+r\frac{\mathrm{d}R}{\mathrm{d}r}=0.$$

In order to integrate this equation, I suppose

$$\frac{a}{r}=1+r_0+r_1\cos 2\tau+e\left(1-\frac{e^2}{8}\right)\cos\xi+\&c.$$

If r be used to denote the terms in r which are found in the elliptic expression, so that

$$\frac{a}{\mathrm{r}}=1+e\left(1-\frac{e^2}{8}\right)\cos\xi+e^2\left(1-\frac{e^2}{3}\right)\cos 2\xi+\frac{9}{8}e^3\cos 3\xi+\frac{4}{3}e^4\cos 4\xi+\&c.,$$

and
$$\frac{a}{r}=\frac{a}{\mathrm{r}}+a\,\delta\frac{1}{r}$$

$$\frac{\mathrm{d}^2 . \mathrm{r}^2}{2\,\mathrm{d}\,t^2} - \frac{\mathrm{d}^2 . \mathrm{r}^3\,\delta\frac{1}{r}}{\mathrm{d}\,t^2} + \frac{3\,\mathrm{d}^2 . \mathrm{r}^4\left(\delta\frac{1}{r}\right)^2}{2\,\mathrm{d}\,t^2} - \frac{2\,\mathrm{d}^2 . \mathrm{r}^5\left(\delta\frac{1}{r}\right)^3}{\mathrm{d}\,t^2}$$

$$-\frac{\mu}{r} + \frac{\mu}{a} + 2\int \mathrm{d}\,R + r\,\frac{\mathrm{d}\,R}{\mathrm{d}\,r} = 0.$$

Let $\mathfrak{r}_n$ be that part of the coefficient of the nth argument in the development of the quantity

$$-\mathrm{r}^3\,\delta\frac{1}{r} + \frac{3}{2}\,\mathrm{r}^4\left(\delta\frac{1}{r}\right)^2 + \&c.,$$

which corresponds to the argument of which n is the index, and let R_n be the coefficient corresponding to the argument of which n is the index in the development of R, R'_n the corresponding coefficient in the development of that part of $\delta\,\mathrm{d}\,R$ which is multiplied by m, and only arises in the second approximation, with its sign changed, then the quantities r_n are given by equations similar to the following,

$$r_1\left\{\left\{1 + 3\,e^2\left(1 + \frac{e^2}{8}\right)\right\}\{2 - 2\,m\}^2 - 1\right\} = (2 - 2\,m)^2\,\mathfrak{r}_1$$

$$-2\left\{\left\{\frac{2}{2 - 2\,m} + 1\right\} m^2\,R_1 + \frac{m^3}{(2 - 2\,m)}\,R'_1\right\}$$

Passing over terms given by M. PLANA and arising from the first approximation with which I agree, I come to r_{22}.

$$r_{22}\left\{\{2 - 2\,m + 3\,\mathrm{c}\}^2 - 1\right\} = (2 - 2\,m + 3\,\mathrm{c})^2\,\mathfrak{r}_{22}$$

$$-2\left\{\frac{2 + 3\,\mathrm{c}}{(2 - 2\,m + 3\,\mathrm{c})} + 1\right\} m^2\,R_{22}$$

$$\mathfrak{r}_{22} = \frac{3}{2}\,r_{10} - \frac{1}{16}\,r_1 \qquad r_{10} = \frac{7}{2}\,m^2 \qquad r_1 = m^2 \qquad {}^*R_{22} = -\frac{25}{32}$$

$$r_{22} = \frac{25 . 3 . 7}{24 . 2 . 2}\,m^2 - \frac{25}{24 . 16}\,m^2 + \frac{2 . 2 . 25}{24 . 32}\,m^2 = \frac{2125}{384}\,m^2$$

M. PLANA has $-\frac{1175}{384}\,m^2$

$$r_{25}\left\{\{2 - m + 2\,\mathrm{c}\}^2 - 1\right\} = (2 - m + 2\,\mathrm{c})^2\,\mathfrak{r}_{25} - 2\left\{\frac{2 + 2\,\mathrm{c}}{2 - m + 2\,\mathrm{c}} + 1\right\} m^2\,R_{25}$$

$$\mathfrak{r}_{25} = \frac{3}{2}\,r_{13} \qquad r_{13} = -\frac{33}{32}\,m^2 \qquad R_{25} = \frac{3}{8}$$

$$r_{25} = -\frac{16 . 3 . 33}{15 . 2 . 32}\,m^2 - \frac{2 . 2 . 3}{15 . 8}\,m^2 = -\frac{7}{4}\,m^2.$$

M. PLANA has $\frac{7}{2}\,m^2$

$$r_{45}\left\{\{2 - m - 3\,\mathrm{c}\}^2 - 1\right\} = \{2 - m - 3\,\mathrm{c}\}^2\,\mathfrak{r}_{45} - 2\left\{\frac{2 - 3\,\mathrm{c}}{2 - m - 3\,\mathrm{c}} + 1\right\} m^2\,R_{45}$$

* Wherever I have found a disagreement with the result of M. PLANA, as this might arise from an error in my development of R, I have verified the terms employed.

$$\mathfrak{r}_{45} = \frac{3}{2} r_{27} - \frac{1}{16} r_7 \qquad r_{27} = \frac{15}{8} m^2 \qquad r_7 = -\frac{m^2}{2} \qquad R_{45} = \frac{7}{64}$$

$$r_{45} = \frac{2.2.8}{3.15} m + \frac{m}{2.16.2} + \frac{2.2.7}{2.64} m = \frac{105}{64} m \qquad \text{M. PLANA has } \frac{285}{64} m$$

$$r_{53} \left\{ \{c + 3m\}^2 - 1 \right\} = (c + 3m)^2 \mathfrak{r}_{53} - 2 \left\{ \frac{c}{c + 3m} + 1 \right\} m^2 R_{53}$$

$$\mathfrak{r}_{53} = \frac{3}{2} r_{35} \qquad r_{35} = -\frac{53}{16} m^2 \qquad R_{53} = \frac{53}{32}$$

$$r_{53} = -\frac{3.53}{6.2.16} m - \frac{2.2.53}{6.32} m = -\frac{127}{64} m \qquad \text{M. PLANA has } -\frac{53}{32} m$$

$$r_{56} \{(c - 3m)^2 - 1\} = (c - 3m)^2 \mathfrak{r}_{56} - 2 \left\{ \frac{c}{c - 3m} + 1 \right\} m^2 R_{56}$$

$$\mathfrak{r}_{56} = \frac{3}{2} r_{35} \qquad R_{56} = \frac{53}{32}$$

$$r_{56} = \frac{3.53}{6.2.16} m + \frac{2.2.53}{6.32} = \frac{371}{192} m \qquad \text{M. PLANA has } \frac{53}{64} m$$

The development of R which I gave*, results from the substitution of the elliptic values of the coordinates of the sun and moon in the disturbing function. The elliptic expression for the radius vector contains no term of which the argument is $\xi - 2\eta$, the longitude (λ') contains the term $+ \frac{3}{4} e \gamma^2 \sin (\xi - 2\eta)$. This is changed when the disturbing function is considered.

$$r_{65} \{(c - 2g)^2 (1 - 3r_0) - 1\} = (c - 2g)^2 \mathfrak{r}_{65} - 2.2.m^2 R_{65}$$

$$\mathfrak{r}_{65} = \frac{3}{2} r_{62} \qquad r_{62} = \frac{m^2}{2} \qquad c = 1 - \frac{3}{4} m^2 \qquad g = 1 + \frac{3}{4} m^2$$

$$r_0 = \frac{m^2}{6} \qquad R_{65} = \frac{9}{8} + \frac{1}{2} r_{65} \qquad (c - 2g)\, 2\, (1 - 3r_0) = 1 + 4m^2$$

$$r_{65} = \frac{3}{4.2.2} - \frac{2.2}{4} \left(\frac{9}{8} + \frac{1}{2} r_{65} \right)$$

$$r_{65} = -\frac{5}{8}.$$

This term, produced by the disturbing force, although independent of m, together with the corresponding term in λ', renders in a certain sense incomplete the coefficients of all terms in my development of R, of which the arguments are any combinations of the quantity $\xi - 2\eta$.

$$r_{73} \{(2 - 3m - 2g)^2 - 1\} = (2 - 3m - 2g)^2 \mathfrak{r}_{73} - 2m^2 R_{73}$$

$$\mathfrak{r}_{73} = 0 \qquad R_{73} = -\frac{21}{16}$$

$$r_{73} = -\frac{2.21}{16} m^2 = -\frac{21}{8} m^2 \qquad \text{M. PLANA has } \frac{7}{8} m - \frac{7}{8} m^2$$

* Philosophical Transactions, 1831, p. 263.

The term above, $-\frac{5}{8} e \gamma^2 \cos(\xi - 2\eta)$, introduces the term $+\frac{3}{4} e \gamma^2 \sin(\xi - 2\eta)$ in the longitude, instead of $-\frac{e\gamma^2}{2} \sin(\xi - 2\eta)$. The terms in R produced in consequence may easily be found from the formula

$$\delta R = -a\left(\frac{\mathrm{d} R}{\mathrm{d} a}\right) \mathrm{r}\, \delta \frac{1}{r} + \frac{\mathrm{d} R}{\mathrm{d} \tau} \delta \lambda$$

taking

$$\mathrm{r}\, \delta \frac{1}{r} = -\frac{5}{8} e \gamma^2 \cos(\xi - 2\eta), \text{ and } \delta\lambda = \frac{5}{4} e \gamma^2 \sin(\xi - 2\eta)$$

and I find that R contains, instead of the terms corresponding to the same arguments given in the Philosophical Transactions, 1831, p. 263.

$$+\frac{15}{32} m^2 e \gamma^2 \cos(2\tau - \xi + 2\tau) - m^2 e \gamma^2 \cos(2\tau + \xi - 2\eta)$$
[68] [69]

$$+\frac{39}{32} m^2 e e_{\prime} \gamma^2 \cos(\xi + \xi_{\prime} - 2\eta) + \frac{105}{64} m^2 e e_{\prime} \gamma^2 \cos(2\tau - \xi - \xi_{\prime} + 2\eta)$$
[83] [86]

$$+\frac{33}{64} m^2 e e_{\prime} \gamma^2 \cos(e\tau + \xi + \xi_{\prime} - 2\eta) - \frac{15}{84} m^2 e e_{\prime} \gamma^2 \cos(2\tau - \xi + \xi_{\prime} + 2\eta)$$
[87] [92]

$$-\frac{231}{64} m^2 e e_{\prime} \gamma^2 \cos(2\tau + \xi - \xi_{\prime} - 2\eta)$$
[93]

The coefficient of arg. 77 is easily found as follows:

$$R = -\frac{r^{\backprime 2}}{4 r_{\prime}^3}\{1 + 3\cos(2\lambda - 2\lambda) - 2 s^2\}.$$

This term can only arise from

$$-\frac{r^2}{4 r_{\prime}^3}\{1 - 3 s^2\}$$

$$= \frac{r^3}{2 r_{\prime}^3} \delta \frac{1}{r} + \frac{3 r^2}{4 r_{\prime}^3} s^2.$$

In which expression it is sufficient to write for $\delta \frac{1}{r}$,

$$-\frac{5}{8} e r^2 \cos(\xi - 2\eta)$$

and to make

$$*s^2 = -\frac{\gamma^2}{2} \cos 2\eta + \gamma^2 e \cos(\xi - 2\eta) + \frac{1}{4} \gamma^2 e^2 \cos(2\xi - 2\eta)$$
[65] [77]

which gives

$$R_{77} = \frac{3.5}{2.2.8} + \frac{3}{4.4} - \frac{3}{4} + \frac{3}{4.2.2.2} = 0.$$

* This is not the expression for s^2 in the elliptic movement: the last term is altered by the disturbing force.

When the elliptic values of the coordinates are substituted in the disturbing function, the term in question arises only from the expansion of the quantity

$$+\frac{3}{4}\frac{r^2}{r_{,}^3}s^2,$$

and in the elliptic motion

$$s^2 = -\frac{\gamma^2}{2}\cos 2\eta + \gamma^2 e \cos(\xi - 2\eta) - \frac{3}{8}\gamma^2 e^2 \cos(2\xi - 2\eta) + \&c.$$

$$R_{77} = -\frac{3.3}{4.8} - \frac{3}{4} + \frac{3}{4.2.2.2} = -\frac{15}{16}.$$

Writing the index between brackets instead of the cosine of the corresponding argument, in order to save space.

$$-\int\frac{dR}{d\lambda}dt$$

$$= \frac{3}{4}\left\{1 - \frac{5}{2}e^2 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2[1] - \frac{9}{2}\left\{1 - \frac{13}{24}e^2 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e[3]$$

$$+\frac{1}{2}\left\{1 - \frac{19}{8}e^2 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e[3] + \frac{21}{8}\left\{1 - \frac{5}{2}e^2 - \frac{123}{56}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e_{,}[6]$$

$$+\frac{3}{8}\left\{1 - \frac{5}{2}e^2 - \frac{e_{,}^2}{8} - \frac{\gamma^2}{2}\right\}m^2 e_{,}[7] - \frac{15}{8}\left\{1 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2}\right\}m e^2[9]$$

$$+\frac{3}{8}\left\{1 - \frac{5}{2}e^2 - \frac{5}{2}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e^2[10] - \frac{63}{4}\left\{1 - \frac{91}{128}e^2 - \frac{123}{56}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e e_{,}[12]$$

$$-\frac{1}{4}\left\{1 - \frac{19}{8}e^2 - \frac{e_{,}^2}{8} - \frac{\gamma^2}{2}\right\}m^2 e e_{,}[13] + \frac{9}{4}\left\{1 - \frac{13}{24}e^2 - \frac{e_{,}^2}{8} - \frac{\gamma^2}{2}\right\}m^2 e e_{,}[15]$$

$$+\frac{7}{4}\left\{1 - \frac{19}{8}e^2 - \frac{123}{56}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e_{,}^2[16] + \frac{51}{8}\left\{1 - \frac{5}{2}e^2 - \frac{115}{51}e_{,}^2 - \frac{\gamma^2}{2}\right\}m^2 e_{,}^2[18]$$

$$+\frac{5}{16}m e^3 - \frac{35}{8}m e_{,}^2 e_{,}[24] - \frac{3}{16}m^2 e^2 e_{,}[25] + \frac{15}{8}m e^2 e_{,}[27] + \frac{21}{16}m^2 e^2 e_{,}[28]$$

$$-\frac{153}{4}m^2 e e_{,}^2[30] + \frac{17}{4}m^2 e e_{,}^2[34] + \frac{845}{64}m^2 e_{,}^3[36] + \frac{m^2 e_{,}^3}{64}[37] + \frac{3}{64}m^2 e^4[39]$$

$$+\frac{9}{32}m^2 e^4[40] + \frac{49}{32}m^2 e^3 e_{,}[42] + \frac{25}{32}m^2 e^3 e_{,}[43] - \frac{7}{32}m^2 e^3 e_{,}[45]$$

$$+\frac{35}{32}m^2 e^3 e_{,}[46] - \frac{255}{32}m^2 e^2 e_{,}^2[48] + \frac{51}{16}m^2 e^2 e_{,}^2[52] - \frac{2535}{64}m^2 e e_{,}^3[54]$$

$$-\frac{1}{96}m^2 e e_{,}^3[55] - \frac{3}{32}m^2 e_{,}^3[57] - \frac{15}{32}m^2 e_{,}^3[58] + \frac{2453}{128}m^2 e_{,}^4[60] - \frac{741}{128}m^2 e_{,}^4[70]$$

I have verified some of the terms in the expression for the reciprocal of the radius vector given by M. PLANA, which depend on $\frac{a^3}{a_{,}^4}$, and arise from the second portion of R; very few, however, of these can be obtained without a further development of the disturbing function, in consequence particularly of the term

$$\frac{5a}{4a_{,}}e_{,}\cos(\tau + \xi_{,}),$$

which is independent of m. In consequence of this term, all the terms in my development of R, of which the arguments are any combination of the quantity $\tau + \xi_{\prime}$ are incomplete.

When the terms depending on γ^2, and those depending on the square of the disturbing force, are neglected, the inequalities of longitude are given by the equation

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{dR}{d\lambda} dt.$$

I find from this equation in λ_4 the term

$$\frac{175}{32} m e^2 = \left\{\frac{405}{32} m e^2 + \frac{15}{4} m e^2 + \frac{15}{8} m e^2\right\}\frac{1}{3},$$

instead of $\frac{195}{32} m^2$, according to M. PLANA*.

λ_6 contains the term $-\frac{1353}{128} m^2 e_{\prime}^2$ instead of $-\frac{1253}{128} m^2 e_{\prime}^2$, for

$$R \ldots\ldots\ldots\ldots + \frac{369}{64} e_{\prime}^3 \cos(2\tau - \xi_{\prime})$$

$$\frac{a}{r} \ldots\ldots\ldots\ldots - \frac{123}{16} m^2 e_{\prime}^3 \cos(2\tau - \xi_{\prime})$$

$$\lambda_6 \ldots\ldots\ldots\ldots \left\{-\frac{123}{8} m^2 - \frac{369}{64} m^2\right\} e_{\prime}^2 = -\frac{1353}{64} m^2 e_{\prime}^2.$$

If the numerical coefficient of the corresponding term in the quantity $-\int \frac{dR}{d\lambda} dt$, be called $\mathfrak{R}$, then I find

$$\lambda_{22} = \left\{2 r_{22} + r_{10} + r_{11} + \frac{9}{8} r_1 + \mathfrak{R}_{22} + \mathfrak{R}_{10} + \frac{5}{4}\mathfrak{R}_4 + \frac{13}{8}\mathfrak{R}_1\right\}\frac{1}{(2 - 2m + 3c)}$$

$$= \left\{\frac{2125}{192} m^2 + \frac{7}{2} m^2 + \frac{33}{16} m^2 + \frac{9}{8} m^2 + \frac{5}{16} m^2 + \frac{3}{8} m^2 + \frac{5}{8} m^2 + \frac{13.3}{8.4} m^2\right\}\frac{1}{5}$$

$$= \frac{779}{192} m^2.$$

M. PLANA has $\frac{1093}{64} m^2$, and for the numerical value of the coefficient converted into sexagesimal seconds $3''\cdot309$. M. DAMOISEAU has $1''\cdot27$; I obtain $\cdot77''$.

I find

$$\lambda_{25} = \left\{2 r_{25} + r_{13} + r_7 + \mathfrak{R}_{25} + \mathfrak{R}_{13} + \frac{5}{4}\mathfrak{R}_7\right\}\frac{1}{(2 - m + 2c)}$$

$$= \left\{-\frac{7}{2} m^2 - \frac{33}{32} m^2 - \frac{m^2}{2} - \frac{3}{16} m^2 - \frac{1}{4} m^2 + \frac{5.3}{4.8}\right\}\frac{1}{4} = -\frac{5}{4} m^2.$$

M. PLANA has $-\frac{95}{64} m^2$, and for the numerical value of the coefficient converted into sexagesimal seconds $-\cdot087$. M. DAMOISEAU has $-\cdot19$; I obtain $-\cdot073$.

* The figures are indistinct in the copy before me.

I find

$$\lambda_{28} = \left\{\frac{49}{2} m^2 + \frac{231}{32} m^2 + \frac{7}{2} m^2 + \frac{21}{16} m^2 + \frac{9}{4} m^2 + \frac{5.21}{4.8} m^2\right\} \frac{1}{4} = \frac{673}{64} m^2.$$

M. PLANA has $\frac{665}{64} m^2$, and for the numerical value of the coefficient converted into sexagesimal seconds $''\cdot607$. M. DAMOISEAU has $''\cdot90$; I obtain $''\cdot615$.

I find

$$\lambda_{49} = \left\{2 r_{49} + r_{31} + r_{19} + \mathfrak{R}_{49} + \mathfrak{R}_{31} + \frac{5}{4} \mathfrak{R}_{19}\right\} \frac{1}{(2 + 2c)} = 0;$$

M. PLANA has $-\frac{885}{64} m$.

I find

$$\lambda_{59} = 2 r_{59} \times \frac{1}{4m} = \frac{591}{16} m, \qquad \text{M. PLANA has } -\frac{77}{32} m;$$

I find

$$\lambda_{60} = \{2 r_{60} + \mathfrak{R}_{60}\} \frac{1}{(2 - 6m)} = -\frac{55}{8} m.$$

These discordances will appear very trifling, considering the nature of the calculations; and it is by no means impossible, after all, that M. PLANA may be right, and that the mistake may be with me, notwithstanding all the pains I have used.

Before the terms in the longitude can be arrived at which depend on γ^2, it is necessary to obtain the expression for the tangent of the latitude s: this may be done by means of the equations

$$\frac{d^2 z}{d t^2} + \frac{\mu z}{r^3} + \frac{m_{\prime} z}{r_{\prime}^3} + \frac{3 m_{\prime} z r' r \cos(\lambda' - \lambda_{\prime})}{r_{\prime}^5} = 0$$

$$z = \frac{r s}{\sqrt{1 + s^2}} \qquad \frac{z}{r} = s - \frac{s^3}{2} + \frac{3}{8} s^5 - \&c.$$

$$s = \frac{z}{r} + \frac{z^3}{2 r^3} - \&c.$$

$$\tan^{-1} s = s - \frac{s^3}{3} + \frac{s^5}{5} - \&c.$$

It is, however, more convenient in the determination of s, to adhere to the method of CLAIRAUT, that is, to the method adopted by M. PLANA, notwithstanding the difficulties which occur in that method, and to which I have before alluded.

The following is the differential equation employed.

$$\left\{\frac{d^2 s}{d \lambda'^2} + s\right\}\left\{1 - \frac{2}{h^2}\int r'^2 \frac{dR}{d\lambda'} d\lambda'\right\}$$
$$+ \frac{r'^2}{h^2}\left\{(1 + s^2)\left(\frac{dR}{ds}\right) - r' s \left(\frac{dR}{dr'}\right) - \left(\frac{dR}{d\lambda'}\right)\frac{ds}{d\lambda'}\right\} = 0$$

Substituting in this equation in the terms multiplied by $m_{\prime}$, for s, $r \sin(g\lambda' - \nu)$, and for $\frac{ds}{d\lambda'}$, $-r\cos(g\lambda' - \nu)$, neglecting the square of the disturbing force and the cube of s, I obtain

$$\frac{d^2 s}{d\lambda^2} + s - \frac{3 r^4}{2 h^2 r_{\prime}^3} \{\sin(2\lambda - 2\lambda_{\prime} - g\lambda + \nu) - \sin(g\lambda - \nu)\}$$

$$- \frac{r^5}{8 h^2 r_{\prime}^4} \Big\{ \frac{21}{2} \cos(\lambda - \lambda_{\prime} - g\lambda + \nu) + \frac{15}{2} \cos(\lambda - \lambda_{\prime} + g\lambda - \nu)$$

$$+ 9 \cos(3\lambda - \lambda_{\prime} - g\lambda + \nu) + 6 \cos(3\lambda - 3\lambda_{\prime} + g\lambda - \nu) \Big\}$$

The simplest method of substituting for $\lambda_{\prime}$ in terms of λ seems to me to be by first obtaining expressions for $\cos\lambda_{\prime}$, $\sin\lambda_{\prime}$, $\cos 2\lambda_{\prime}$, $\sin 2\lambda_{\prime}$, in terms of $n_{\prime} t$. Having obtained these expressions, they may be reduced to terms of λ by LAGRANGE's theorem*; but when the higher powers of m are neglected, it is sufficient to write $m\lambda$ instead of $n_{\prime} t$.

$$\sin 2\lambda_{\prime} = (1 - 4 e_{\prime}^2) \sin 2 n_{\prime} t - 2 e \sin(2 n_{\prime} t - \xi_{\prime}) + 2 e_{\prime} \sin(2 n_{\prime} t + \xi_{\prime})$$

$$+ \frac{3}{4} e_{\prime}^2 \sin(2 n_{\prime} t - 2\xi_{\prime}) + \frac{13}{4} e_{\prime}^2 \sin(2 n_{\prime} t + 2\xi_{\prime}) + \&c.$$

$$\cos 2\lambda_{\prime} = (1 - 4 e_{\prime}^2) \cos 2 n_{\prime} t - 2 e_{\prime} \cos(2 n_{\prime} t - \xi_{\prime}) + 2 e_{\prime} \cos(2 n_{\prime} t + \xi_{\prime})$$

$$+ \frac{3}{4} e_{\prime}^2 \cos(2 n_{\prime} t - 2\xi_{\prime}) + \frac{13}{4} e_{\prime}^2 \cos(2 n_{\prime} t + 2\xi_{\prime}) + \&c.$$

Great facility results in the following substitutions in consequence of the coefficients being alike in the corresponding arguments of the expressions $\sin\lambda_{\prime}$, $\cos\lambda_{\prime}$; $\sin 2\lambda_{\prime}$, $\cos 2\lambda_{\prime}$, &c.; so

$$\sin(2\lambda - 2\lambda_{\prime} - g\lambda) = (1 - 4 e^2) \{\sin(2\lambda - g\lambda) \cos 2 m\lambda - \cos(2\lambda - g\lambda) \sin 2 m\lambda\}$$

$$- 2 e_{\prime} \{\sin(2\lambda - g\lambda) \cos(2 m\lambda - c_{\prime} m\lambda) - \cos(2\lambda - g\lambda) \sin(2 m\lambda - c_{\prime} m\lambda)\}$$

$$+ 2 e_{\prime} \{\sin(2\lambda - g\lambda) \cos(2 m\lambda + c_{\prime} m\lambda) - \cos(2\lambda - g\lambda) \sin(2 m\lambda + c_{\prime} m\lambda)\}$$

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* By LAGRANGE's theorem, if

$$u = \theta - f\theta$$

$$\theta = u + fu + \frac{d.(fu)^2}{2\, du} + \frac{d^2 (fu)^3}{2.3\, du^2} + \&c.$$

So

$$a_{\prime}^3 r_{\prime}^{-3} = 1 + \frac{3}{2} e_{\prime}^2 \left(1 + \frac{5}{4} e_{\prime}^2\right) + 3 e_{\prime} \left(1 + \frac{9}{8} e_{\prime}^2\right) \cos\xi_{\prime}$$

$$+ \frac{9}{2} e_{\prime}^2 \left(1 + \frac{7}{9} e_{\prime}^2\right) \cos 2\xi_{\prime} + \frac{53}{8} e_{\prime}^3 \cos 3\xi_{\prime} + \frac{77}{8} e_{\prime}^4 \cos 4\xi_{\prime}$$

$$m\lambda = n_{\prime} t + 2 m e \left(1 - \frac{e^2}{8}\right) \sin\xi + \frac{5}{4} m e^2 \left(1 - \frac{11}{30} e^2\right) \sin 2\xi$$

$$+ \frac{13}{12} m e^3 \sin\xi + \frac{103}{96} m e^4 \sin 4\xi + \&c.$$

Hence evidently

$$a_{\prime}^3 r_{\prime}^3 = 1 + \frac{3}{2} e_{\prime}^2 \left(1 + \frac{5}{4} e_{\prime}^2\right) + 3 e_{\prime} \left(1 + \frac{9}{8} e_{\prime}^2\right) \cos(c_{\prime} m\lambda)$$

$$+ \frac{9}{2} e_{\prime}^2 \left(1 + \frac{7}{9} e_{\prime}^2\right) \cos(2 c_{\prime} m\lambda) + \frac{53}{8} e_{\prime}^3 \cos(3 c_{\prime} m\lambda) + \frac{77}{9} e_{\prime}^4 \cos(4 c_{\prime} m\lambda)$$

+ terms multiplied by m, $c_{\prime}$ may be considered as equal to unity.

$$+\frac{3}{4} e_{,}^{2} \{\sin (2\lambda - g\lambda) \cos (2 m\lambda - 2 c_{,} m\lambda) - \cos (2\lambda - g\lambda) \sin (2 m\lambda - 2 c_{,} m\lambda)\}$$

$$+\frac{13}{4} e_{,}^{2} \{\sin (2\lambda - g\lambda) \cos (2 m\lambda + 2 c_{,} m\lambda) - \cos (2\lambda - g\lambda) \cos (2 m\lambda + 2 c_{,} m\lambda)\}$$

$$= (1 - 4 e_{,}^{2}) \sin (2\lambda - 2 m\lambda - g\lambda)$$

$$- 2 e_{,} \sin (2\lambda - 2 m\lambda + c_{,} m\lambda - g\lambda) + 2 e_{,} \sin (2\lambda - 2 m\lambda - c_{,} m\lambda - g\lambda)$$

$$+\frac{3}{4} e_{,}^{2} \sin (2\lambda - 2 m\lambda + 2 c_{,} m\lambda - g\lambda) + \frac{13}{4} e_{,}^{2} \sin (2\lambda - 2 m\lambda - 2 c_{,} m\lambda - g\lambda)$$

$$\frac{m_{,} r^{4}}{h^{2} r_{,}^{3}} = m^{2} \left\{1 + \frac{2}{3} e_{,}^{2} + 3 e_{,} \cos c_{,} m\lambda + \frac{9}{2} e_{,}^{2} \cos 2 c_{,} m\lambda\right\}$$

$$\{1 + 2 e^{2} - 4 e \cos c\lambda + 5 e^{2} \cos 2 c\lambda\}$$

$$\frac{3}{2} \frac{m_{,} r^{4}}{h^{2} r_{,}^{3}} = m^{2} \left\{\frac{3}{2} + \frac{9}{4} e_{,}^{2} + 3 e^{2} - 6 e_{,} \cos c\lambda + \frac{9}{2} e_{,} \cos c_{,} m\lambda + \frac{15}{2} e^{2} \cos 2 c\lambda\right.$$

$$\left. - 9 e_{,} \cos (e\lambda - c_{,} m\lambda) - 9 e e_{,} \cos (e\lambda + c_{,} m\lambda) + \frac{27}{4} e_{,}^{2} \cos 2 c_{,} m\lambda\right\}$$

All the terms which I have verified in the expression for the latitude in terms of the true longitude agree with those given by M. PLANA.

If $\lambda = n t + \alpha$, then by TAYLOR's theorem,

$$s = (s) + \left(\frac{d s}{d\lambda}\right) \alpha + \left(\frac{d^{2} s}{d\lambda^{2}}\right) \frac{\alpha^{2}}{2} + \left(\frac{d^{3} s}{d\lambda^{3}}\right) \alpha^{3} + \&c.$$

(s) being the quantity arising from the substitution of $n t$ for λ, in the expression for s in terms of λ. In this manner I found the same terms as those given by M. PLANA, except $-\frac{15}{32} m e^{2} \gamma \sin (2\tau - 2\xi + \eta)$ instead of $-\frac{15}{64} m e^{2} \gamma \sin (2\tau - 2\xi + \eta)$, and $-\frac{3}{8} m e e_{,} \gamma \sin (2\tau + \xi + \xi_{,} - \eta)$ instead of $\frac{1}{2} m e e_{,} \gamma \sin (2\tau + \xi + \xi_{,} - \eta)$.

I next obtained s^{2}, in order to procure the terms in the longitude depending on γ^{2}. The quantity e in my notation does not accord with that quantity in the work of M. PLANA, but with $e\left(1 - \frac{\gamma^{2}}{4}\right)$; so that, in order to arrive at the same figures in some of the terms multiplied by γ^{2}, this circumstance must be attended to.

I find

$$\lambda_{70} = \left\{-\frac{39}{16} m^{2} - \frac{11}{16} m^{2} - \frac{33}{32} m^{2} - m^{2} - \frac{1}{4} m^{2} - \frac{3}{8} m^{2}\right\} \frac{1}{5} = -\frac{37}{32} m^{2}$$

instead of $-\frac{39}{32} m^{2}$, according to M. PLANA; and I find

$$\lambda_{85} = -\left\{\frac{35}{8} m + \frac{7}{8} m - \frac{35}{16} m - \frac{7}{8} m - \frac{7}{16} m\right\} = -\frac{7}{4} m$$

instead of $\frac{7}{8} m$, according to M. PLANA.

These are the only discrepancies which I have noted in the terms multiplied by γ^2 which I have examined. In making use of the development of R before alluded to,

$$1-\frac{\gamma^2}{2}+\frac{7}{16}\gamma^4 \text{ is to be substituted for } \cos^4\frac{\gamma}{2}.$$

The relation between the constants h and a is to be obtained from the equations

$$r^4\frac{\mathrm{d}\lambda^2}{\mathrm{d}t^2}=h^2-2\int r^2\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}\lambda$$

$$\frac{r^2\mathrm{d}\lambda^2+\mathrm{d}r^2}{\mathrm{d}t^2}-\frac{2\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}R=0;$$

where, for simplicity, I have neglected at present quantities depending on γ^2.

$$\mathrm{d}\,.\frac{1}{r}=-\frac{\mathrm{d}r}{r^2}$$

$$\mathrm{d}r^2=r^4\left(\mathrm{d}\,.\frac{1}{r}\right)^2=\left(\frac{1}{r}\right)^{-4}\left(\mathrm{d}\,.\frac{1}{r}\right)^2=a^2(1-4r_0)\left\{\frac{e^2c^2n^2}{2}+\frac{e^2r_3^2n^2}{2}\right\}$$

$2\int\mathrm{d}R$ can give no constant term *; therefore, considering the constant part only of the equation above, since the coefficients corresponding to all the arguments must be separately identical,

$$h^2\left\{1+2r_0+\frac{e^2}{2}+\frac{e^2r_3^2}{2}\right\}=a\mu\left\{1+2r_0-(1-4r_0)\right\}\frac{e^2c^2}{2}+\frac{e^2r_3^2}{2}\right\}$$

r_4^2, being already multiplied by m^4, is not to be taken into account in this approximation.

$$c^2\{1-3r_0\}-1+2m^2=0 \qquad\qquad c^2=1-\frac{7}{2}m^2$$

$$h^2\left\{1-m^2+\frac{e^2}{2}+\frac{33}{128}m^2e^2-\frac{3}{2}m^2e_{\prime}^2\right\}=a\mu\left\{1-\frac{e^2}{2}-m^2-\frac{321}{128}m^2e^2-\frac{3}{2}m^2e_{\prime}^2\right\}$$

$$h^2=a\mu\left\{1-\frac{e^2}{2}-m^2-\frac{321}{128}m^2e^2-\frac{3}{2}m^2e_{\prime}^2\right\}$$

$$\left\{1+m^2-\frac{e^2}{2}-\frac{33}{128}m^2e^2+\frac{3}{2}m^2e_{\prime}^2-m^2e^2\right\}$$

$$=a\mu\left\{1-e^2-\frac{241}{64}m^2e^2\right\}$$

$$h=a\mu\left\{1-\frac{e^2}{2}-\frac{241}{128}m^2e^2\right\}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}=\frac{h}{r^2}-\frac{1}{r^2}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\mathrm{d}t.$$

* This seems at variance with the equation of M. PLANA, vol. i. p. 122,

$$-\frac{2a}{\sigma}\int\mathrm{d}'\Omega=-\frac{m^2}{2}\left\{1+\frac{1}{2}\gamma^2+\frac{3}{2}e^2+\frac{3}{2}e^2\gamma^2\cos(2\varpi-2\theta)+\&c.\right\}$$

which equation I am unable to understand.

The second term on the right-hand side of this equation gives no constant quantity: hence

$$\frac{h}{r^2} = \sqrt{\frac{\mu}{a^3}} \left\{ 1 - \frac{e^2}{2} - \frac{241}{128} m^2 e^2 \right\} \left\{ 1 - m^2 + \frac{e^2}{2} + \frac{33}{128} m^2 e^2 - \frac{3}{2} m^2 e_{\prime}^2 \right\}$$

$$\lambda = \sqrt{\frac{\mu}{a^3}} \left\{ 1 - m^2 - \frac{9}{8} m^2 e^2 - \frac{3}{2} m^2 e_{\prime}^2 \right\} t + \&c.$$

If the constant quantity which multiplies t be called

$$\mathrm{n} = \sqrt{\frac{\mu}{\mathrm{a}^3}}$$

$$\sqrt{\frac{\mu}{a^3}} \left\{ 1 - m^2 - \frac{9}{8} m^2 e^2 - \frac{3}{2} m^2 e_{\prime}^2 \right\} = \sqrt{\frac{\mu}{\mathrm{a}^3}}$$

$$\mathrm{a} = a \left\{ 1 + \frac{2}{3} m^2 + \frac{3}{4} m^2 e^2 + m^2 e_{\prime}^2 \right\}$$

$$\frac{a}{r} = \frac{\mathrm{a} \left\{ 1 + \frac{2}{3} m^2 + \frac{3}{4} m^2 e^2 + m^2 e_{\prime}^2 \right\}^{-1}}{r} = 1 - \frac{m^2}{2} - \frac{3}{4} m^2 e^2 - \frac{3}{4} m^2 e_{\prime}^2.$$

Reverting to the equation

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{h}{r^2} - \frac{1}{r^2} \int \frac{\mathrm{d}R}{\mathrm{d}\lambda} \mathrm{d}t.$$

The second term of this equation gives no term multiplied by $\cos \xi$; therefore,

$$\mathrm{d}\lambda = \mathrm{n}\,\mathrm{d}t + \frac{2h}{a^2} (1 + r_0)\, e \cos \xi\, \mathrm{d}t$$

$$= \mathrm{n}\,\mathrm{d}t + 2\mathrm{n} \left(1 - \frac{m^2}{2}\right) (1 + m^2)\, e \cos \xi\, \mathrm{d}t$$

$$c = 1 - \frac{7}{4} m^2 \qquad \mathrm{c}\,\mathrm{n} = c n \qquad \mathrm{c} = 1 - \frac{3}{4} m^2$$

$$\lambda = \mathrm{n}t + 2 \left(1 + \frac{3}{4} m^2 + \frac{m^2}{2}\right) e \sin \xi$$

$$\lambda = \mathrm{n}t + 2 \left(1 + \frac{5}{4} m^2\right) e \sin \xi + \&c.$$

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} + \frac{m^2 e_{\prime}^2}{4} + e \left(1 + \frac{2}{3} m^2\right) \cos \xi + \&c.$$

These are the expressions for λ and for $\frac{1}{r}$, when the quantity e is retained; but if the coefficient of $\sin \xi$, in the expression for λ, be called $2\,\mathrm{e}$, after the manner adopted for the planets in the first volume of the Mécanique Céleste, so that

$$\lambda = \mathrm{n}t + 2\,\mathrm{e} \sin \xi + \&c., \quad \text{then}$$

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} + \frac{m^2 e_{\prime}^2}{4} + \mathrm{e} \left(1 - \frac{7}{12} m^2\right) \cos \xi + \&c.$$

As the preceding results do not quite agree with those of M. PLANA, I shall endeavour to show how they may be obtained from the same equations which he employs.

$$\frac{d^2 . \frac{1}{r}}{d \lambda^2} + \frac{1}{r} - \frac{1}{h^2}\left\{1 + 2\int r^2 \frac{d R}{d \lambda} d \lambda\right\} - \frac{r}{h^2}\left\{r \frac{d R}{d r} - \frac{1}{r}\frac{d R}{d \lambda} \cdot \frac{d r}{d \lambda}\right\} = 0$$

$$R = -\frac{r^2}{4 r_{\prime}^3}\left\{1 + 3 \cos (2 \lambda - 2 \lambda_{\prime})\right\}$$

In order to integrate the preceding differential equation, let

$$\frac{h^2}{r} = \mu \{1 + r_0 + r_1 \cos (2 \lambda - 2 m \lambda) + e \cos (c \lambda - \varpi) + e r_3 \cos (2 \lambda - 2 m\lambda - c\lambda + \varpi)\}$$

[0] [1] [2] [3]

The letters r_0, r_1, &c., being now used in a somewhat different sense to heretofore, having now reference to the expression for $\frac{1}{r}$ in terms of the true longitude.

$$\frac{1}{r}^{-3} = \frac{h^6}{\mu^3}(1 + 3 e^2 + \&c.)$$

Neglecting $e_{\prime}^2$,

$$r^0 + \frac{h^6}{2 a_{\prime}^3} \frac{m_{\prime}}{\mu}(1 + 3 e^2) = 0.$$

Substituting in this equation for h its elliptic value which is allowable, $r_0 = -\frac{m^2}{2}$, also $r_3 = \frac{15}{8} m$,

$$\left(\frac{1}{r}\right)^{-2} = \frac{h^4}{\mu^2}\left\{1 + m^2 + \frac{225}{128} \times 3 m^2 e^2 + \frac{3 e^2}{2} + 3 m^2 e^2\right\}$$

$$= \frac{h^4}{\mu^2}\left\{1 + \frac{3 e^2}{2} + m^2 + \frac{1059}{128} m^2 e^2\right\}$$

$$\frac{d t}{d \lambda} = \frac{r^2}{h} = \frac{h^3}{\mu^2}\left\{1 + \frac{3 e^2}{2} + m^2 + \frac{1059}{128} m^2 e^2\right\} = \sqrt{\frac{a^3}{\mu}}$$

$$h^2 = a\left\{1 + \frac{3 e^2}{2} + m^2 + \frac{1059}{128} m^2 e^2\right\}^{-\frac{2}{3}}$$

$$= a\left\{1 - e^2 - \frac{2}{3} m^2 - \frac{739}{192} m^2 e^2\right\}$$

$$= a\left\{1 - e^2 - \frac{2}{3} m^2 - \frac{739}{192} m^2 e^2\right\}\left\{1 + \frac{2 m^2}{3} + \frac{3}{4} m^2 e^2\right\}$$

$$= a\left\{1 - e^2 - \frac{241}{64} m^2 e^2\right\} \text{ as before.}$$

$$\frac{h^2}{r} = \frac{\mathrm{a}\left\{1 - e^2 - \frac{2}{3}\,m^2 - \frac{739}{192}\,m^2\,e^2\right\}}{r}$$

$$= 1 - \frac{m^2}{2} + e\cos\left(\mathrm{c}\,\lambda - \varpi\right) + \&\mathrm{c}.$$

$$\frac{\mathrm{a}}{r} = 1 + e^2 + \frac{m^2}{6} + \frac{899}{192}\,m^2\,e^2 + e\left(1 + \frac{2}{3}\,m^2\right)\cos\left(\mathrm{c}\,\lambda - \varpi\right) + \&\mathrm{c}.$$

My letter a appears to correspond with a $(1 + p)$, or a in M. PLANA's notation.

n	n
c	c
μ	σ
R	Ω
$\frac{1}{r}$	u
λ	v.

Putting $e\left((1 - \frac{m^2}{2}\right)$ instead of e in the various expressions found above,

$$\lambda = \mathrm{n}\,t + 2\left(1 + \frac{3}{4}\,m^2\right)e\sin\xi,$$

which then so far agrees with the expression of M. PLANA*, and I then find

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} + \frac{m^2 e_{\prime}^2}{4} + e\left(1 + \frac{m^2}{6}\right)\cos\xi + \&\mathrm{c}.$$

$$\frac{\mathrm{a}}{r} = 1 + e^2 + \frac{m^2}{6} + \frac{707}{192}\,m^2\,e^2 + e\left(1 + \frac{m^2}{6}\right)\cos\left(\mathrm{c}\,\lambda - \varpi\right).$$

M. PLANA has

$$\frac{\mathrm{a}}{r} = 1 + \frac{m^2}{6} - \frac{45}{16}\,m^2\,e^2 + \frac{m^2 e_{\prime}^2}{4} + e\left(1 + \frac{m^2}{6}\right)\cos\xi + \&\mathrm{c}.\dagger$$

$$\frac{\mathrm{a}}{r} = 1 + e^2 + \frac{m^2}{6} + \frac{167}{192}\,m^2\,e^2 + e\left(1 + \frac{m^2}{6}\right)\cos\left(\mathrm{c}\,\lambda - \varpi\right) + \&\mathrm{c}.\ddagger$$

which equations do not agree with those I have found. I am, however, well aware how difficult it is to escape error in these inquiries, and wish to be understood as not offering any of the results contained in this paper too confidently.

Wherever I presumed to have arrived at figures differing from those of M. PLANA, I verified afresh all the steps of the process contained in previous papers, particularly the corresponding term in the development of R. Thus I have found by means of the expressions given §, that three times the numerical coefficient of $e_{\prime}^3\cos\left(2\,\tau - \xi_{\prime}\right)$

* Vol. i. p. 574. † Vol. i. p. 664. ‡ Vol. i. p. 636.

§ Philosophical Transactions, 1832, p. 601.

in the development of R,

$$= \frac{63}{16} + \frac{81}{64} + \frac{45}{16} + \frac{27}{32} - \frac{153}{16} + \frac{18}{32} + \frac{30}{8} + \frac{30}{32} + \frac{204}{16} = \frac{1107}{64}.$$

The coefficient in question $= \frac{369}{64} = \frac{21}{8} \times \frac{153}{56}$.

But I have found that the following corrections are required. For

$$+ \frac{3}{8} \left\{ 1 - \frac{5}{2} e^2 - 4 e_,^2 \right\} \cos^4 \frac{\iota}{2} \frac{a^2}{a_,^3} e_, \cos (2 \tau + \xi_,) \qquad [7]$$

read

$$+ \frac{3}{8} \left\{ 1 - \frac{5}{2} e^2 - \frac{e_,^2}{8} \right\} \cos^4 \frac{\iota}{2} \frac{a^2}{a_,^3} e_, \cos (2 \tau + \xi_,)$$

and for

$$- \frac{51}{64} \frac{a^2}{a_,^3} e_,^2 \gamma^2 \cos (2 \xi_, - 2 \eta) - \frac{45}{64} \frac{a^2}{a_,^3} e_,^2 \gamma^2 \cos (2 \xi_, + 2 \eta) - \frac{195}{64} \frac{a^2}{a_,^3} e_,^2 \gamma^2 \cos (2 \tau - 2 \xi_, - 2 \eta)$$

[95] [96] [97]

read

$$- \frac{27}{16} \frac{a^2}{a_,^3} e_,^2 \gamma^2 \cos (2 \xi_, - 2 \eta) - \frac{27}{16} e_,^2 \gamma^2 \cos (2 \xi_, + 2 \eta) - \frac{51}{16} \frac{a^2}{a_,^3} e_,^2 \gamma^2 \cos (2 \tau - 2 \xi_, - 2 \eta).$$